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Electrical & Computer ENGINEERING

# A Family of Provably Correct Algorithms for Exact Triangle Counting

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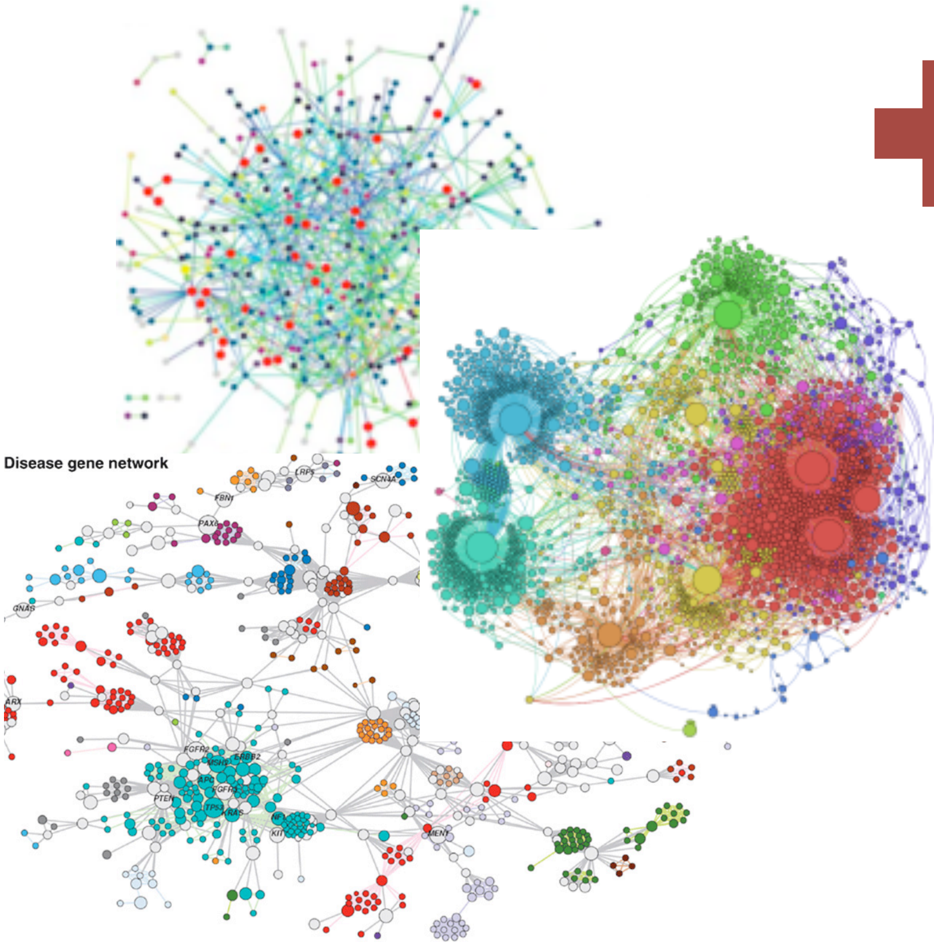
Correctness 2017

# Motivation

## Graphs are everywhere




https://en.wikipedia.org/wiki/Bioinformatics
http://www.mkbergman.com/968/a-new-best-friend-gephi-for-large-scale-networks/
https://www.cs.umd.edu/research/projects/16672

## Correctness in HPC

$$C = AA^T + C$$

# FLAME

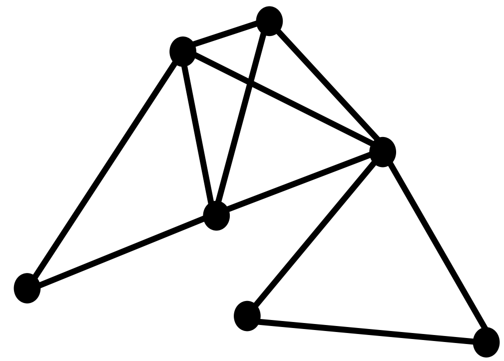
- Formal Linear Algebra Methods Environment
- Main components
  - 8-step algorithm derivation methodology
    - Requires loop invariants as input
    - Produces both algorithm and proof of correctness
  - Index-free APIs for implementing derived algorithms
- libFLAME
  - Formally derived common LAPACK functionality
  - High performance

**Key Idea: Find Loop Invariants for Graph Algorithms**

# Specifying the problem

- Number of triangles in a graph

$$\Delta = \frac{1}{6}\Gamma(A^3) \quad A \rightarrow \begin{pmatrix} A_{TL} & A_{TR} \\ A_{TR}^T & A_{BR} \end{pmatrix}$$

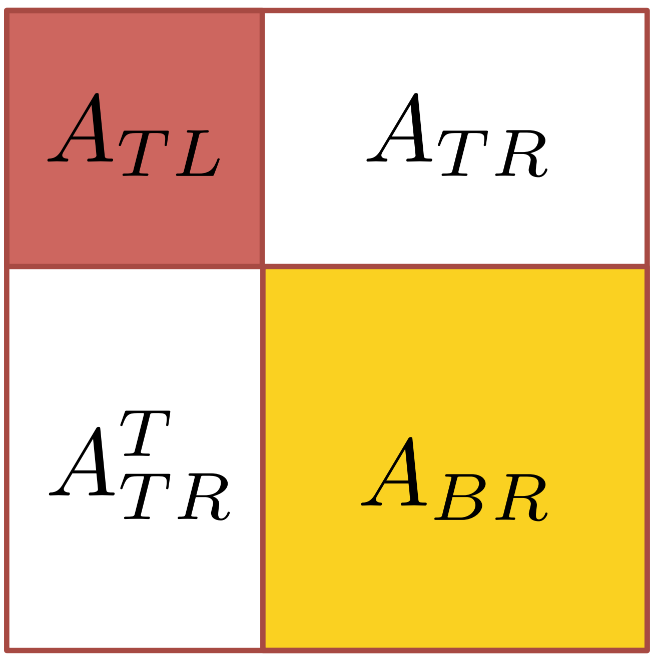


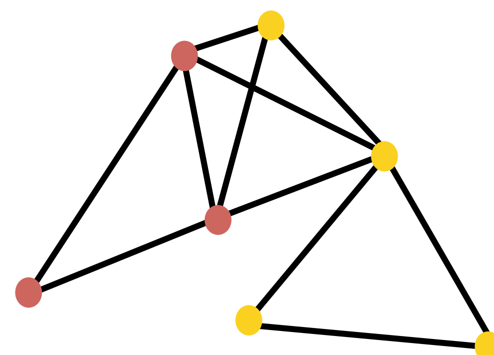
$$\Delta = \frac{1}{6}\Gamma(A_{TL}^3) + \frac{1}{2}\Gamma(A_{TR}^T A_{TL} A_{TR}) + \frac{1}{2}\Gamma(A_{TR} A_{BR} A_{TR}^T) + \frac{1}{6}\Gamma(A_{BR}^3)$$

**Partitioned Matrix Expression (PME)**

# Partitioned Matrix Expression (PME)

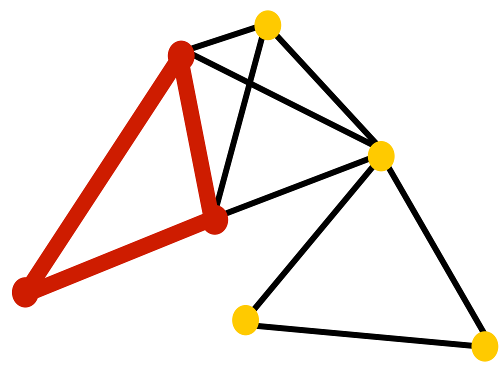
$$\Delta = \frac{1}{6}\Gamma(A_{TL}^3) + \frac{1}{2}\Gamma(A_{TR}^T A_{TL} A_{TR}) + \frac{1}{2}\Gamma(A_{TR} A_{BR} A_{TR}^T) + \frac{1}{6}\Gamma(A_{BR}^3)$$



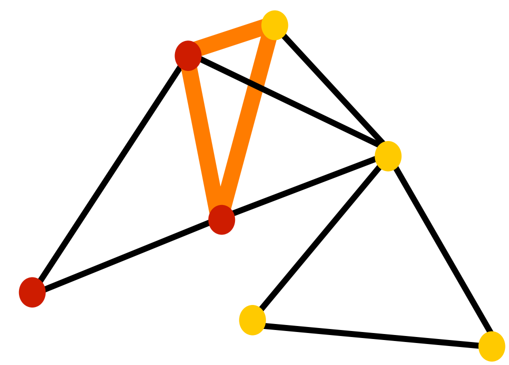



# Different Types of Triangles

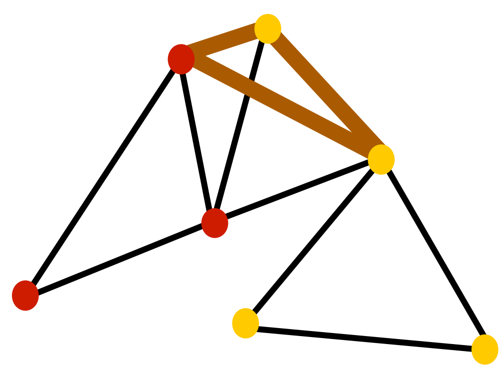
$$\Delta = \frac{1}{6}\Gamma(A_{TL}^3) + \frac{1}{2}\Gamma(A_{TR}^T A_{TL} A_{TR}) + \frac{1}{2}\Gamma(A_{TR} A_{BR} A_{TR}^T) + \frac{1}{6}\Gamma(A_{BR}^3)$$



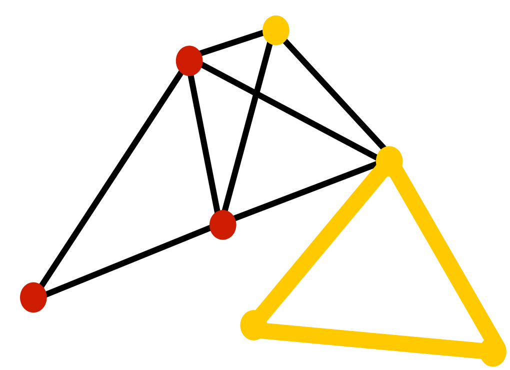
Category I



Category II



Category III



Category IV

# Finding Loop Invariants

- Assertion that must be true at start and end of every iteration

```
while ( G ) {



}
```

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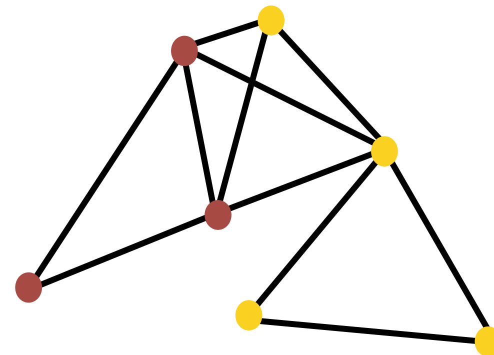

**Loop Invariant:**
**# of triangles that have been computed**

# Loop Invariants from PME

$$\Delta = \underbrace{\frac{1}{6}\Gamma(A_{TL}^3)}_{\text{I}} + \underbrace{\frac{1}{2}\Gamma(A_{TR}^T A_{TL} A_{TR})}_{\text{II}} + \underbrace{\frac{1}{2}\Gamma(A_{TR} A_{BR} A_{TR}^T)}_{\text{III}} + \underbrace{\frac{1}{6}\Gamma(A_{BR}^3)}_{\text{IV}}$$

$$\Delta = \frac{1}{6}\Gamma(A_{TL}^3) + {\color{lightgray}\frac{1}{2}\Gamma(A_{TR}^T A_{TL} A_{TR}) + \frac{1}{2}\Gamma(A_{TR} A_{BR} A_{TR}^T) + \frac{1}{6}\Gamma(A_{BR}^3)}$$

$$\Delta = \frac{1}{6}\Gamma(A_{TL}^3) + {\color{lightgray}\frac{1}{2}\Gamma(A_{TR}^T A_{TL} A_{TR}) +} \frac{1}{2}\Gamma(A_{TR} A_{BR} A_{TR}^T) {\color{lightgray}+ \frac{1}{6}\Gamma(A_{BR}^3)}$$

$$\Delta = \frac{1}{6}\Gamma(A_{TL}^3) + \frac{1}{2}\Gamma(A_{TR}^T A_{TL} A_{TR}) {\color{lightgray}+ \frac{1}{2}\Gamma(A_{TR} A_{BR} A_{TR}^T) + \frac{1}{6}\Gamma(A_{BR}^3)}$$

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$$\Delta = {\color{lightgray}\frac{1}{6}\Gamma(A_{TL}^3) + \frac{1}{2}\Gamma(A_{TR}^T A_{TL} A_{TR}) + \frac{1}{2}\Gamma(A_{TR} A_{BR} A_{TR}^T)} + \frac{1}{6}\Gamma(A_{BR}^3)$$

$$\Delta = {\color{lightgray}\frac{1}{6}\Gamma(A_{TL}^3) + \frac{1}{2}\Gamma(A_{TR}^T A_{TL} A_{TR}) +} \frac{1}{2}\Gamma(A_{TR} A_{BR} A_{TR}^T) + \frac{1}{6}\Gamma(A_{BR}^3)$$

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# Derived Algorithms

Algorithm: $\hat{\Delta} := \frac{1}{6}\Gamma(\hat{A}^3)$

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{TR}^T & A_{BR} \end{array}\right)$$

where $A_{TL}$ is a $0 \times 0$ matrix

while $m(A_{TL}) < m(A)$ do

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{TR}^T & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{01}^T & \alpha_{11} & a_{12}^T \\ \hline A_{02}^T & a_{12} & A_{22} \end{array}\right)$$

where $\alpha_{11}$ is a $1 \times 1$ matrix

| **Algorithm 1** | **Algorithm 2** |
|---|---|
| $\Delta := \Delta + \frac{1}{2}a_{01}^T A_{00} a_{01}$ | $\Delta := \Delta + a_{01}^T A_{02} a_{21}$ |
| **Algorithm 3** | **Algorithm 4** |
| $\Delta := \Delta + \frac{1}{2}a_{01}^T A_{00} a_{01}$<br>$\Delta := \Delta + \frac{1}{2}a_{12}^T A_{22} a_{12}$<br>$\Delta := \Delta - a_{01}^T A_{02} a_{21}$ | $\Delta := \Delta + \frac{1}{2}a_{12}^T A_{22} a_{12}$ |

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{TR}^T & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{01}^T & \alpha_{11} & a_{12}^T \\ \hline A_{02}^T & a_{12} & A_{22} \end{array}\right)$$

endwhile

Algorithm: $\hat{t} := \frac{1}{6}\Gamma(A^3)$

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{TR}^T & A_{BR} \end{array}\right)$$

where $A_{BR}$ is a $0 \times 0$ matrix

while $m(A_{TL}) < m(A)$ do

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{TR}^T & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{01}^T & \alpha_{11} & a_{12}^T \\ \hline A_{02}^T & a_{12} & A_{22} \end{array}\right)$$

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| **Algorithm 5** | **Algorithm 6** |
|---|---|
| $\Delta := \Delta + \frac{1}{2}a_{01}^T A_{00} a_{01}$ | $\Delta := \Delta + a_{01}^T A_{02} a_{21}$ |
| **Algorithm 7** | **Algorithm 8** |
| $\Delta := \Delta + \frac{1}{2}a_{01}^T A_{00} a_{01}$<br>$\Delta := \Delta + \frac{1}{2}a_{12}^T A_{22} a_{12}$<br>$\Delta := \Delta - a_{01}^T A_{02} a_{21}$ | $\Delta := \Delta + \frac{1}{2}a_{12}^T A_{22} a_{12}$ |

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endwhile

# Derived Algorithms

**Algorithm:** $\tilde{\Delta} := \frac{1}{6}\Gamma(\hat{A}^3)$

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**while** $m(A_{TL}) < m(A)$ **do**

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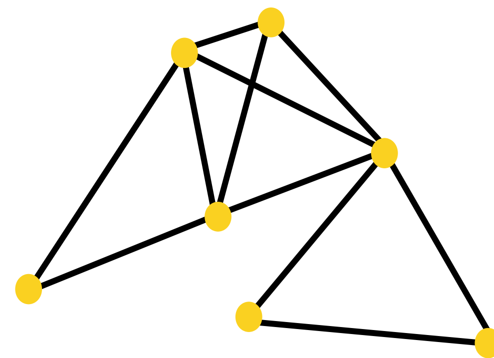
$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{TR}^T & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ a_{01}^T & \alpha_{11} & a_{12}^T \\ \hline A_{02}^T & a_{12} & A_{22} \end{array}\right)$$

**endwhile**



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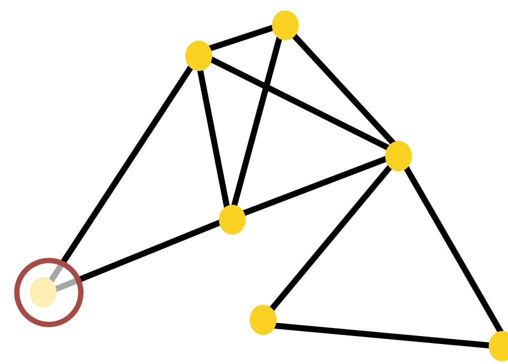

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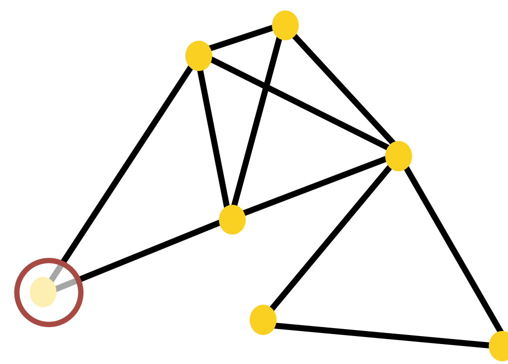
**endwhile**



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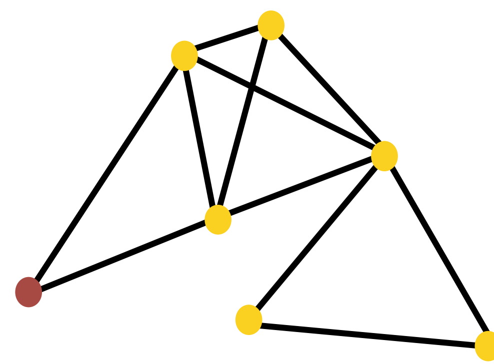

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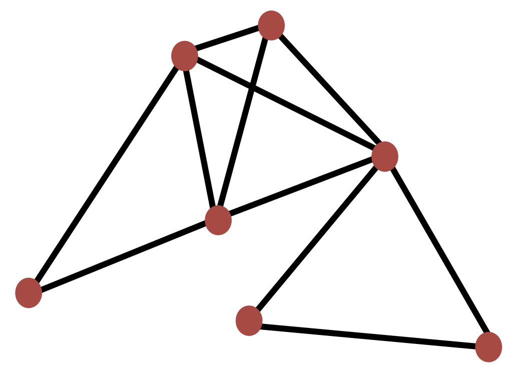
endwhile



# FLAME API

- Index-free API for implementing derived algorithms

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{TR}^T & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{01}^T & \alpha_{11} & a_{12}^T \\ \hline A_{02}^T & a_{12} & A_{22} \end{array}\right)$$

where $\alpha_{11}$ is a $1 \times 1$ matrix

```
FLA_Repart_2x2_to_3x3( ATL, /**/ ATR,       &A00, /**/ &a01, &A02,
                    /* ************* */ /* ************************** */
                                           &a10t, /**/ &alpha11, &a12t,
                      ABL, /**/ ABR,        &A20, /**/ &a21, &A22,
                      1, 1, FLA_BR );
```

- Separation of implementation and algorithm concerns

# Extension to the FLAME API

- Existing API supports only dense matrices
- Introduced
  - Support for sparse matrices (CSR)
  - Additional function, `dist_to_nonzero_column`
    - Conceptually treat sparse matrices as dense
    - Each “dense” block has only one non zero column
    - Returns blocking parameter to the next non zero


Actual Layout

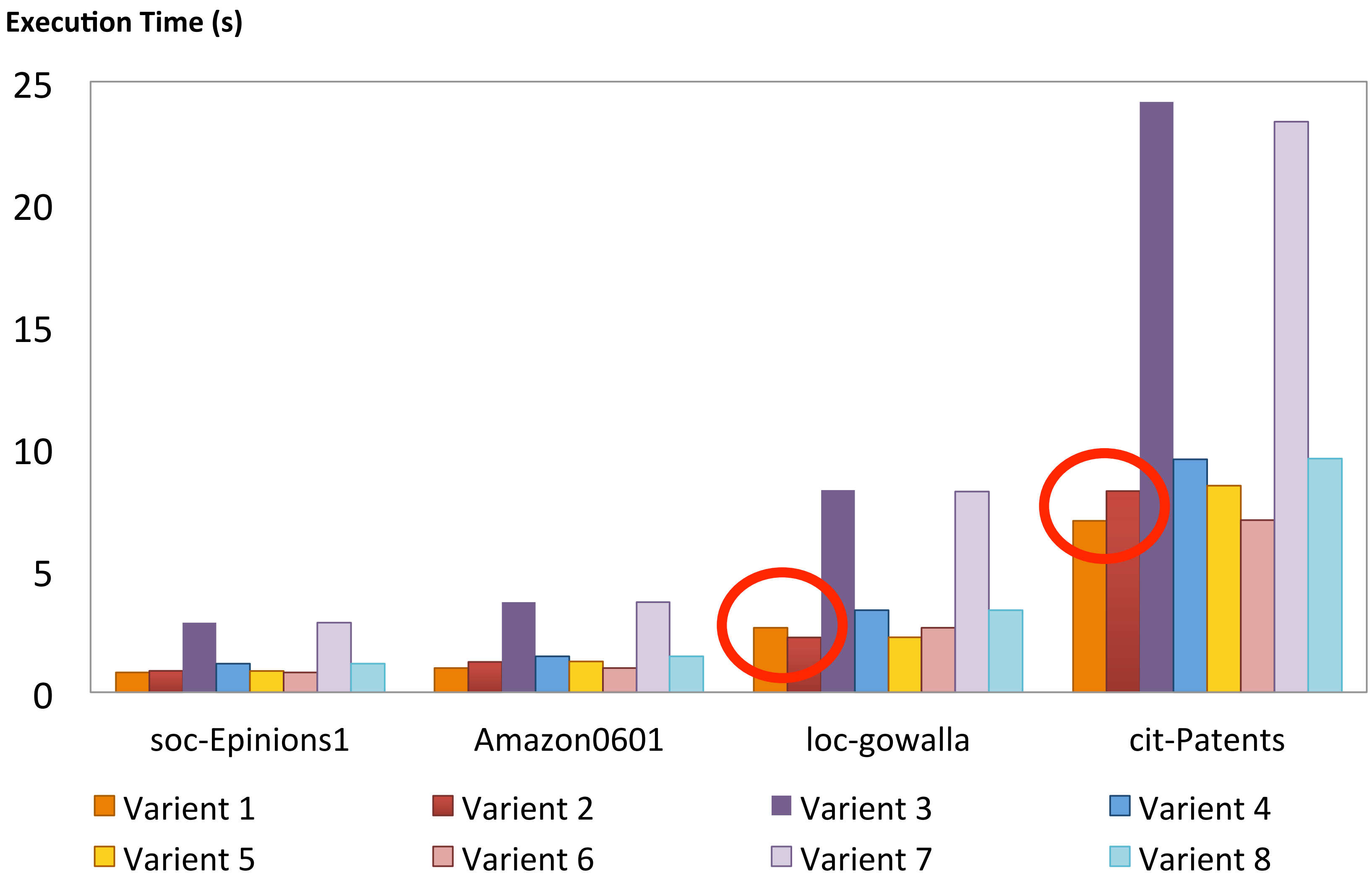
Conceptual Layout

# Datasets

- Datasets
  - Graph Challenge website
  - Stanford Large Network Dataset (SNAP)
  - Directed graphs were made undirected

| Dataset | Nodes | Edges | Triangles |
|---|---|---|---|
| soc-Epinions1 | 75,879 | 508,837 | 1,624,481 |
| Amazon0601 | 403,394 | 3,387,388 | 3,986,507 |
| loc-gowalla | 196,591 | 950,327 | 2,273,138 |
| Cit-Patents | 3,774,768 | 16,518,948 | 7,515,023 |
| Com-Friendster | 65,608,366 | 1,806,067,135 | 4,173,724,142 |

# Performance




Sequential Performance on Intel i7 E5-2667 v 3 Haswell , 3.2GHz

# Summary

- A family of formally derived algorithms for computing triangles in a graph
- First extension of the FLAME methodology beyond DLA
- API to support CSR format
- To do:
  - Analyze graph features that determine performance of algorithm
  - Reduce overhead of indexing functions

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# Questions?

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